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A HEURISTIC METHOD OF SOLVING NONLINEAR PROBLEMS OF THE ELLIPTIC TYPE FOR DOUBLY CONNECTED REGIONS

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The proposed method is based on the use of certain information of a qualitative or quantitative character about the properties of quantities appearing in a nonlinear boundary value problem for its formulation, and thus to simplify the original problem. Such information can be provided by solutions of similar problems, experimental data, and by some other considerations. The extent of simplification evidently depends on the available information and the manner of its application. The approximate solution obtained with the use of such information is called here the zero approximation. The proposed iteration method differs essentially from known methods by the procedure of determining the zero approximation, and in the iteration process algorithm. It reduces a problem in partial derivatives to the solution of an ordinary differential equation for each approximation, which considerably reduces the amount of computations as compared to that required by the method of finite differences and finite elements. The zero approximation derived by the proposed method is often equivalent to two or three approximations obtained by the method of small parameter (depending on the magnitude of the latter). The question of the method convergence is not considered here, but the theorem about the iteration process uniform convergence to the solution is proved.

1. Let us consider an example of the most effective method of information introduction in the initial statement of a boundary value problem. We consider for simplicity the elliptic type nonlinear equation

$$D^* (U_{xx},\ U_{xy},\ U_{yy},\ U_x,\ U_y,\ U,\ x,\ y) = 0 \tag{1.1}$$

where D^* is an analytic function of eight arguments and U is the unknown function dependent on coordinates x and y. The solution of Eq. (1.1) is sought in the doubly connected region Ω bounded by two curves Γ_1 and Γ_2 in the plane (x, y) whose equations are of the form $\xi_{01}(x, y) = \alpha_1$ and $\xi_{02}(x, y) = \alpha_2$, respectively. Let the boundary conditions for Eq. (1.1) be of the form

$$U (x,\ y) |_{\Gamma_i} = v_i = \text{const}, \quad i = 1, 2 \tag{1.2}$$

Let us assume that the form of curves $U (x, y) = \text{const}$ is a priori known and based on some considerations. We define the curves on which the unknown function is constant by the approximate equation of the form

$$\xi = \xi (x,\ y) \tag{1.3}$$

Since in conformity with (1.2) function U is constant along the boundaries Γ_1 and

Γ_2, hence

$$\xi(x, y)|_{\Gamma_i} = \alpha_i = \text{const} \tag{1.4}$$

Equalities (1.4) can be taken as the equations of boundaries Γ_1 and Γ_2, respectively. In boundary value problems the form of function (1.3) is often known only at the boundaries Γ_1 and Γ_2, i.e.

$$\xi|_{\Gamma_i} = \xi_{0i}(x, y) = \alpha_i \tag{1.5}$$

It is reasonable to assume that within the Ω-region function $\xi(x, y)$ varies in the interval $[\alpha_1, \alpha_2]$ in some continuous manner. The simplest approximate definition of such function which would satisfy condition (1.4) in accordance with (1.5) is given by the relationship

$$[\xi_{01}(x, y) - \alpha_1](\xi - \alpha_2) = [\xi_{02}(x, y) - \alpha_2](\xi - \alpha_1) \tag{1.6}$$

which can be extended by taking into account additional information about function $\xi(x, y)$.

We are, thus, considering the case in which the information about the behavior of the unknown function $U(x, y)$ allows an approximate formulation of the equation for the set of curves (in this case $\xi = \text{const}$) along which function U varies little.

We pass in the problem (1.1), (1.2) from variables (x, y) to variables (ξ, η) of which the variables $\eta = \eta(x, y)$ may be arbitrary but such that the Jacobian of transformation $(x, y) \rightarrow (\xi, \eta)$ is nonzero in region $\bar{\Omega}$. Equation (1.1) and boundary conditions (1.2) expressed in new variables are of the form

$$D(U_{\xi\xi}, U_{\xi\eta}, U_{\eta\eta}, U_\xi, U_\eta, U, \xi, \eta) = 0 \tag{1.7}$$
$$U = v_i \quad \text{for} \quad \xi = \alpha_i, \quad i = 1, 2$$

If the differnetial equation (1.7) does not explicitly contain variable η, then variable ξ in (1.3) is self-similar and the problem reduces to solving an ordinary differential equation. In the opposite case function U varies along curves $\xi(x, y) = \text{const}$. However according to the method of introduction of variable ξ in (1.3) the variation of U along the curve $\xi = \text{const}$ must be small so that derivatives U_η, $U_{\xi\eta}$ and $U_{\eta\eta}$ can be negelcted in Eq. (1.7) when deriving the zero approximation, since it is small in comparison with other terms. For the determination of the zero approximation $U_0(\xi, \eta)$ we thus obtain a nonlinear differential equation of second order with two boundary conditions

$$D_0(U_{0\xi\xi}, 0, 0, U_{0\xi}, \quad 0, U_0, \xi, \eta) = 0 \tag{1.8}$$
$$U_0 = v_i \quad \text{for} \quad \xi = \alpha_i, \quad i = 1, 2$$

In the integration of Eq. (1.7) the variable η plays the part of a parameter and, consequently, the approximation U_0 in (1.8) depends on the two variables ξ and η.

We denote function $\xi(x, y)$ in (1.3) by $\xi_0(x, y)$, since it was used in the determination of the zero approximation. Having determined $U_0(\xi_0, \eta)$, it is reasonable to assume that the more exact equation of the curve along which the sought solution of $U(x, y)$ does not vary is of the form $\xi = U_0(\xi_0, \eta)$, hence for ξ_0 we substitute the new variable

$$\xi_1 = U_0(\xi_0, \eta) \tag{1.9}$$

Substituting variables $(x, y) \to (\xi_1, \eta)$ and repeating the reasoning similar to that used after the introduction of variables when deriving the zero approximation, for the first approximation we obtain, similarly to (1.8), the nonlinear second order ordinary differential equation

$$D_1(U_{1\xi\xi}, 0, 0, U_{1\xi}, 0, U_1, \xi_1, \eta) = 0 \tag{1.10}$$
$$U_1 = v_i \quad \text{for } \xi_1 = r_i, i = 1, 2$$

To obtain the k-th approximation it is necessary to make the substitution $(x, y) \to (\xi_k, \eta)$, where

$$\xi_k = U_{k-1}(\xi_{k-1}, \eta) \tag{1.11}$$

After transformation (1.11) region $\bar{\Omega}$ becomes a doubly connected set $\bar{\Omega}_k(\xi_k, \eta)$, where $\bar{\Omega}$ is the closure of region Ω. Since problems of convergence are not considered here, we can assume that the doubly connected set $\{\bar{\Omega}_k\}$ has the doubly connected set $\bar{\Omega}^*$ as its limit. We set the partial derivatives with respect to the variable η of the unknown function $U_k(\xi_k, \eta)$ equal zero

$$\frac{\partial U_k}{\partial \eta} = \frac{\partial^2 U_k}{\partial \eta^2} = \frac{\partial^2 U_k}{\partial \xi_k \partial \eta} = 0 \tag{1.12}$$

Equation (1.1) in partial derivatives is then again of the form of a nonlinear second order ordinary differential equation

$$D_k(U_{k\xi_k\xi_k}, 0, 0, U_{k\xi_k}, 0, U_k, \xi_k, \eta) = 0 \tag{1.13}$$

with boundary conditions (1.14)

$$U_k = v_i \quad \text{for } \xi_k = r_i, \ i = 1, 2 \tag{1.14}$$

Note that because assumption (1.12) is not strictly satisfied, function U_k is only an approximate solution of problem (1.1),(1.2). If some approximation U_n proves to be dependent on the variable ξ_n and independent of η, then assumption (1.12) is strictly satisfied for U_n, and function $U_n(\xi_n)$ is then an exact solution of the input problem. In that case Eq. (1.13) for U_n does not contain η.

For a uniformly convergent iteration process the following relationship is satisfied:

$$\lim\ [U_k(\xi_k, \eta) - \xi_k] = 0, \quad k \to \infty \tag{1.15}$$

We denote the limit function of sequence $\{\xi_k\}$ by ξ^*, and shall determine whether the limit ξ^* is the solution of the input problem. For this we shall prove the following theorem. If in the described above iteration process the sequence of functions $\{U_k\}$ with all of their first and second derivatives with respect to variables ξ_k and η uniformly converge in the doubly connected region $\bar{\Omega}$ to some function ξ^* which is twice continuously differentiable with respect to the totality of variables, then that limit function is the solution of the input problem (1.1),(1.2).

To prove this it is sufficient to verify that function ξ^* exactly satisfies assumption (1.12), i.e.

$$\frac{\partial \xi^*}{\partial \eta} = \frac{\partial^2 \xi^*}{\partial \xi^* \partial \eta} = \frac{\partial^2 \xi^*}{\partial \eta^2} = 0 \qquad (1.16)$$

The last of equalities (1.16) is evident if the first of these is satisfied. According to the conditions of the theorem it is possible to differentiate equality (1.15) with respect to variable η, and then, taking into account that ξ_k and η are independent variables, obtain

$$\lim dU_k(\xi_k, \eta) / d\eta = 0, \quad k \to \infty \qquad (1.17)$$

Since $\lim dU_k / d\eta = d\xi^* / d\eta$ when $k \to \infty$, hence (1.17) provides the proof of equalities (1.16).

Partial differentiation of equality (1.15) with respect to variable ξ_k successively once and twice shows that the approximation U_k for the convergent process has the property

$$\lim \frac{\partial U_k(\xi_k, \eta)}{\partial \xi_k} = 1, \quad \lim \frac{\partial^2 U_k}{\partial \xi_k^2} = 0, \ k \to \infty \qquad (1.18)$$

The indicated algorithm for obtaining an approximate solution of problem (1.1), (1.2) admits modifications. Thus, instead of (1.11) another method of defining variable ξ_k is possible, for example

$$\xi_{k+1} = \nu_k U_k(\xi_k, \eta) + (1 - \nu_k)\xi_k, \ 1 \geqslant \nu_k(x, y) > 0, \ k \geqslant 1 \qquad (1.19)$$

Formula (1.19) is unsuitable for defining ξ_1 when $k = 0$, since the definitions of functions ξ_0 and ξ_k $(k \geqslant 1)$ are essentially different. Hence it is possible to define ξ_1 as follows:

$$\xi_1 = \nu_0 U_0(\xi_0, \eta) + (1 - \nu_0) U_0(\xi_0, \eta^*), \ 1 \geqslant \nu_0(x, y) > 0 \qquad (1.20)$$

where η^* is some characteristic value of variable η which corresponds to region $\bar{\Omega}$. A direct test shows that the uniform convergence of the iteration in $\bar{\Omega}$ implies the uniform convergence of function U_k in $\bar{\Omega}$ to the same limit function. Hence the theorem on the convergence of approximations to the solution remains valid in the case of (1.19).

2. Let us consdier as an example the problem of pure shear of visco-plastic material between two cylindrical surfaces [1] in the absence of rigid zones in the flow region. In a rectangular Cartesian system of coordinates (x, y, z) the surfaces of the stationary Γ_1 and moving Γ_2 plates are defined by equations

$$y = y_0 = h \ (\Gamma_1), \quad y = y_1 = h\delta \cos x \ (\Gamma_2)$$

The mobile plate translates at constant velocity v_0 parallel to the z-axis. In the case of small amplitudes δ this problem can be solved by the method of the small parameter, in which the solution of the Couette problem corresponds to the zero approximation. We shall solve the problem by the method described in Sect. 1 without making any assumptions about the magnitude of parameter δ.

We assume in this problem that each particle has a single component of velocity U parallel to the z-axis, i.e.

$$u_z = U(x, y), \quad u_x = u_y = 0$$

Among components of the strain rate tensor the nonzero components are

$$\gamma_x = \partial U/\partial x, \quad \gamma_y = \partial U/\partial y$$

We relate velocity U to v_0, stresses τ_x and τ_y to the yield stress k_0, the coordinates x and y to the quantity h, and the rates of shear γ_x and γ_y to the quantity v_0/h. The relation between stresses τ_x and τ_y, and the rate of strain γ_x and γ_y for a viscoplastic medium is then of the form

$$\tau_x = G\gamma_x, \quad \tau_y = G\gamma_y, \quad G = (\gamma_x^2 + \gamma_y^2)^{-1/2} + \eta_0 v_0/k_0 h \tag{2.1}$$

where η_0 is the viscosity coefficient. The substitution of (2.1) into the equation of equilibrium yields a nonlinear second order equation in partial derivatives with boundary conditions (on the assumption that particles of the medium adhere to the rigid boundaries)

$$\frac{\partial}{\partial x}\left(G\frac{\partial U}{\partial x}\right) + \frac{\partial}{\partial y}\left(G\frac{\partial U}{\partial y}\right) = 0 \tag{2.2}$$

$$U|_{\Gamma_1} = 0, \quad U|_{\Gamma_2} = 1$$

Several examples of particular solutions of boundary value problems of viscoplastic material shear [2, 3] can be quoted, which show that lines of equal velocities are of a form that is close to that of the flow region boundary. In the immediate vicinity of Γ_1 and Γ_2 the lines $U = \text{const}$ almost exactly follows the contour of these boundaries. Because of this we define ξ and η by formulas

$$y = y_0 + \xi(y_1 - y_0), \qquad x = \eta \tag{2.3}$$

$$\xi = (1 - y)/(1 - \delta\cos x), \quad 1 \geqslant \xi \geqslant 0$$

Transformation (2.3) is such that lines $\xi = \text{const}$ are cosine curves similar to line Γ_2 with their amplitude dampened with decreasing distance from Γ_1. When $\delta < 1$, transformation (2.3) is nondegenerate throughout the region comprised between Γ_1 and Γ_2.

Note that the method of the small parameter requires the more severe restriction $\delta \ll 1$.

Carrying out the substitution $(x, y) \to (\xi, \eta)$ and omitting the derivatives of U with respect to variable η, we obtain from (2.2) the nonlinear second order differential equation with boundary conditions

$$[1 + (\xi\delta\sin x)^2]\, V_\xi + \xi(\delta\sin x)^2 V = 0, \quad V = GU_{0\xi} \tag{2.4}$$

$$\xi = 0, \ U_0 = 0; \ \xi = 1, \ U_0 = 1$$

where the variable x appears as a parameter. Integrating this equation and determining the constants of integration by using boundary conditions, we obtain for this problem the zero approximation solution

$$U_0 = I(\xi, x)/I(1, x)$$

$$I(\xi, x) = \ln\left[\xi\delta\sin x + \sqrt{1 + (\xi\delta\sin x)^2}\right]$$

At the limit $\delta \to 0$ (2.4) becomes the solution of the Couette problem. The solution of problem (2.2) by the method of the small parameter, taking δ for the latter, and taking into account the zero, first, and second approximations, entails a considerable amount of computation, and is not adduced here owing to its unwieldiness.

Numerical values of coordinates along which velocity U is equal 0.5 are tabulated below

x	0	45	90	120	180
y_0	0.5	0.5	0.5	0.5	0.5
y_1	0.5364	0.5263	0.5000	0.4792	0.4562
y_2	0.5413	0.5296	0.5015	0.4820	0.4634
y^*	0.5403	0.5286	0.5012	0.4829	0.4656
y	0.5419	0.5298	0.5014	0.4819	0.4628

Coordinates y_0, y_1, and y_2 of line $U = 0.5$ were obtained by the method of the small parameter in the zero, first, and second approximations, respectively; y^* denotes coordinates determined by the first iteration of the proposed here method, and y relates to the limit value of these coordinates determined by the third iteration of this method.

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REFERENCES

1. Freudenthal, A.M. and Heuringer, H., Mathematical Theories of Inelastic Continuous Medium. Moscow, Fizmatgiz, 1962.

2. Miasnikov, V.P., Some exact solutions for rectilinear motion of a viscoplastic medium. PMTF, No.2, 1961.

3. Bykovtsev, G.I. and Chernyshov, A.D., On viscoplastic flows in non-circular cylinders in the presence of a pressure differential. PMTF, No. 4, 1964.

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